Turbulent mixing: causality, nonlocality, & compressibility

- Extracting concepts from grand challenge DNS
- Learn more about the theory
- Insightful surprises
  - Test-field method
  - Integral kernels

Brandenburg, Rädler, & Schrinner (2008, A&A 482, 732)

Hubbard & Brandenburg (2009, ApJ 706, 712)

MIXING TIMESCALES IN A SUPERNOVA-DRIVEN INTERSTELLAR MEDIUM

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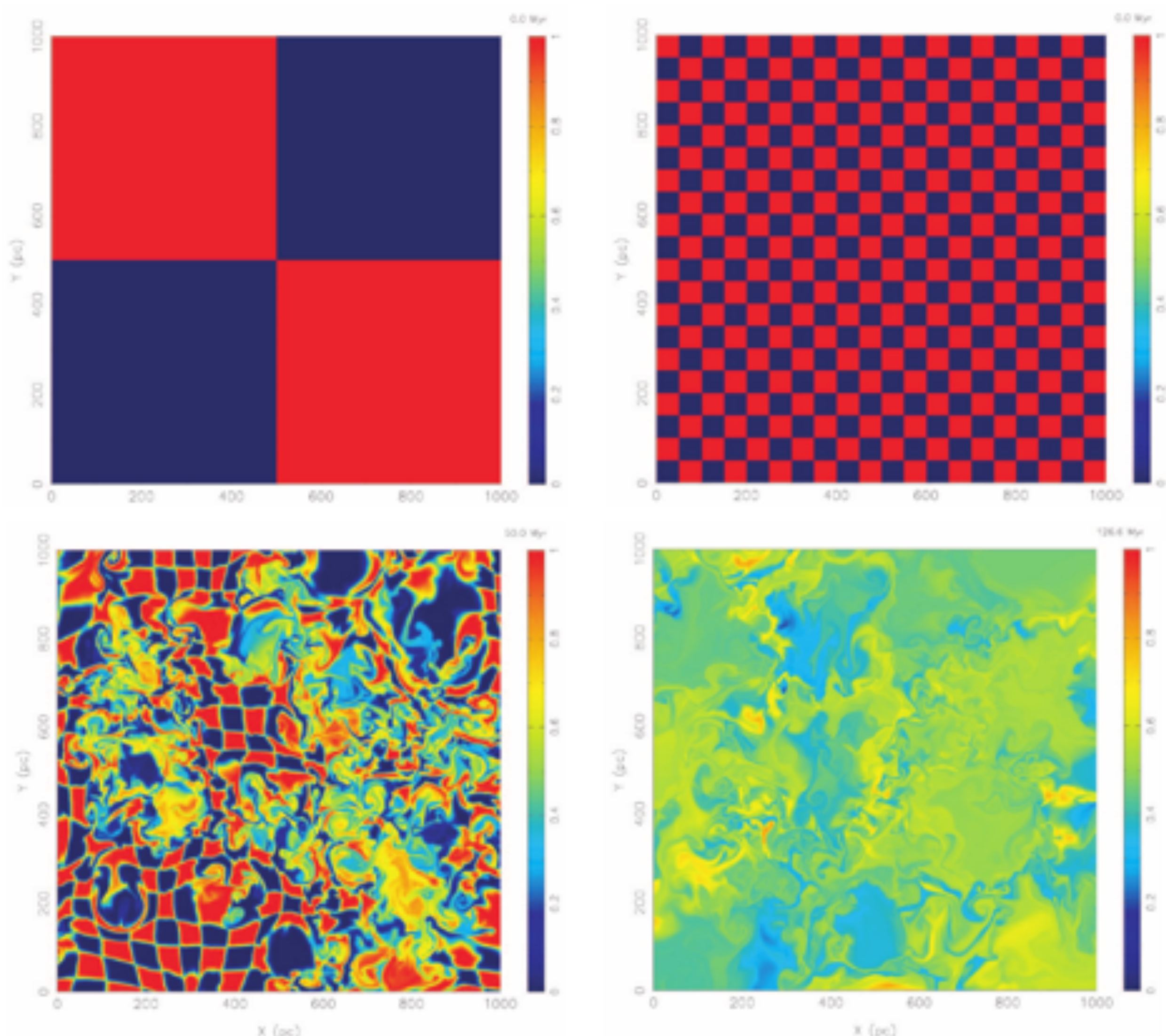

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Run[a]	Δx[b] (pc)	σ/σ_{gal}[c]	τ_{25}[d] (Myr)	τ_{50}[d] (Myr)	τ_{500}[d] (Myr)
L11	5	1	...	303.1	311.2
L12	5	5	...	176.2	192.1
L13	5	10	...	143.0	167.2
L14	5	15	...	132.2	152.7
L15	5	20	...	123.5	144.9
L16	5	30	...	108.4	130.5
L17	5	50	...	107.0	127.2
L21	2.5	1	306.3	318.8	326.5
L22	2.5	5	180.9	193.8	216.6
			147.2	162.4	184.5
			139.6	144.9	159.2
			130.9	138.5	151.6

$$\tau^{-1} = \kappa_{zz} k^2 \propto l^{-\delta} \propto k^{\delta}$$

$$\kappa_{zz} \propto k^{-2+\delta}$$

Test-field method

assume
Here 1-D:
$$\overline{C} = \overline{C}(z,t)$$

ansatz:
$$\mathsf{F}_z = \gamma_z \overline{C} - \kappa_{zz} \partial_z \overline{C}$$

$$\overline{C}^s = \sin kz, \qquad \overline{C}^c = \cos kz$$

$$\mathsf{F}_z^C = \gamma_z \cos kz + \kappa_{zz} k \sin kz$$
$$\mathsf{F}_z^S = \gamma_z \sin kz - \kappa_{zz} k \cos kz$$

$$\begin{pmatrix} \gamma_z \\ -\kappa_{zz} k \end{pmatrix} = \begin{pmatrix} \cos kz & \sin kz \\ -\sin kz & \cos kz \end{pmatrix} \begin{pmatrix} \overline{\mathsf{F}_z^C} \\ \overline{\mathsf{F}_z^S} \end{pmatrix}$$

Convolution → multiplication in *k* space

$$\overline{\mathcal{E}}_i(z) = \int \left[\hat{\alpha}_{ij}(\zeta)\overline{B}_j(z-\zeta) - \hat{\eta}_{ij}(\zeta)\overline{J}_j(z-\zeta)\right] \mathrm{d}\zeta, \tag{12}$$

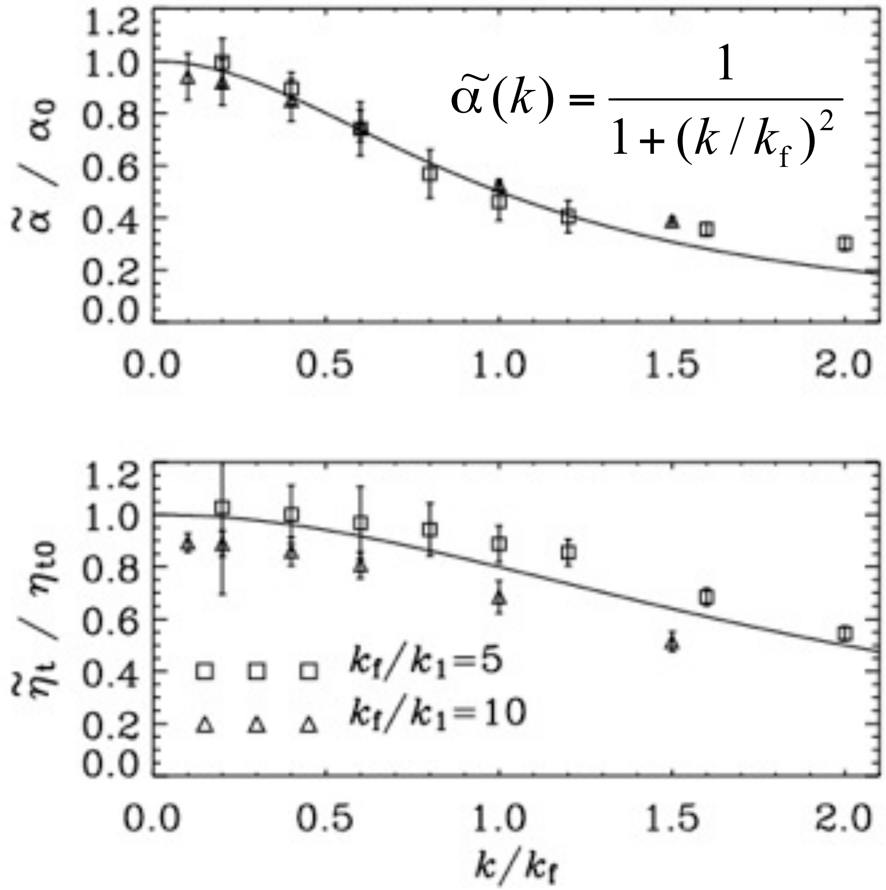
and instead of (11),

$$\tilde{\overline{\mathcal{E}}}_i(k) = \tilde{\alpha}_{ij}(k)\tilde{\overline{B}}_j(k) - \tilde{\eta}_{ij}(k)\tilde{\overline{J}}_j(k), \tag{13}$$

with real $\hat{\alpha}_{ij}(\zeta)$ and $\hat{\eta}_{ij}(\zeta)$, which are even in ζ, and real $\tilde{\alpha}_{ij}(k)$ and $\tilde{\eta}_{ij}(k)$, which are even in k. A justification of these relations is given in Appendix A. We have further

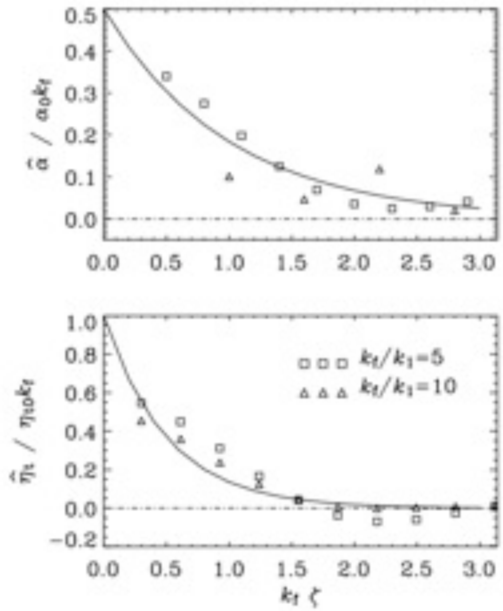
$$\tilde{\alpha}_{ij}(k) = \int \hat{\alpha}_{ij}(\zeta) \cos k\zeta \,\mathrm{d}\zeta, \quad \tilde{\eta}_{ij}(k) = \int \hat{\eta}_{ij}(\zeta) \cos k\zeta \,\mathrm{d}\zeta. \tag{14}$$

Here for magnetic diffusivity



$$\tilde\alpha(k) = \frac{1}{1+(k/k_f)^2}$$

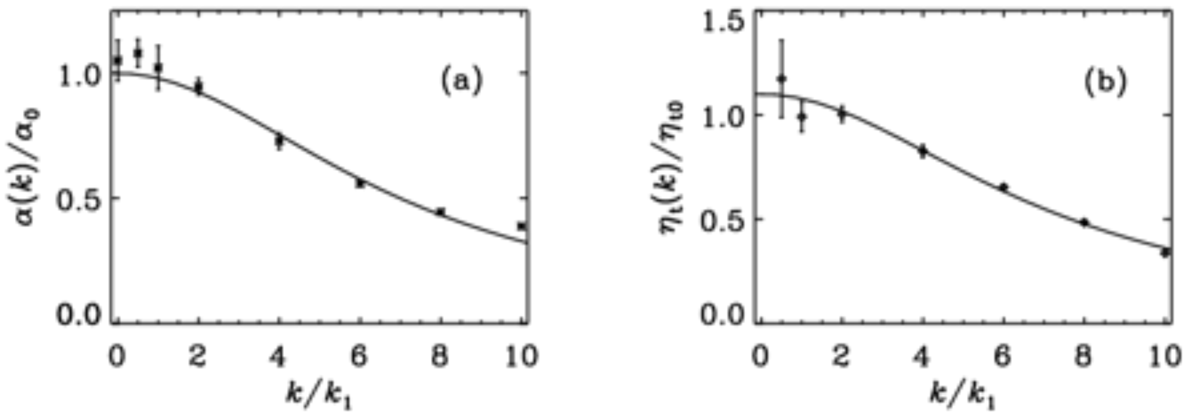
$$\alpha\overline{\mathbf{B}} \rightarrow \int \hat\alpha(z-z')\,\overline{\mathbf{B}}(z')\,\mathrm{d}z'$$



$$\hat\alpha(\zeta) = \exp(-k_f\zeta)$$

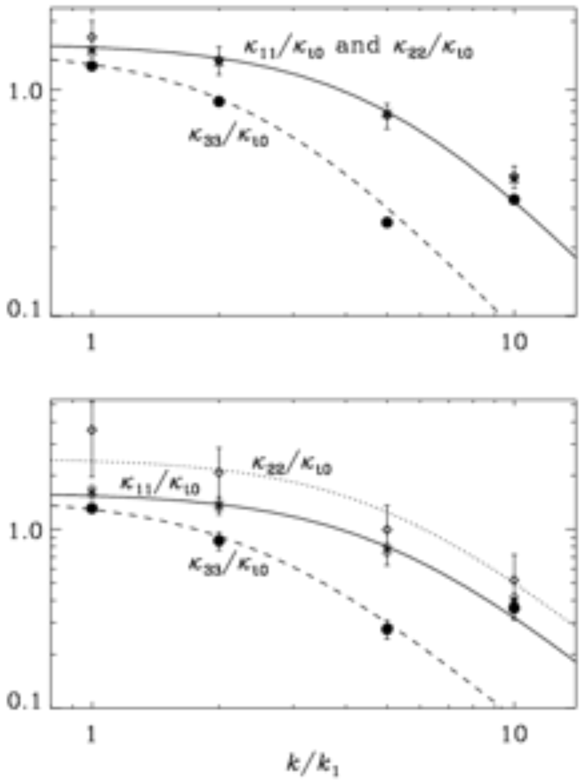
$$\hat\alpha(z) = \int \tilde\alpha(k)\,e^{ikz}\,\frac{\mathrm{d}k}{2\pi}$$

Confirmed also for other cases



Mitra et al.
(2009 A&A 495, 1)

U=(0, Sx, 0)
Sh=S/uk=-0.13



$\sim 1/[1+(k/k_f)^2]$ seems now compulsory
Significant when MFM would produce SS

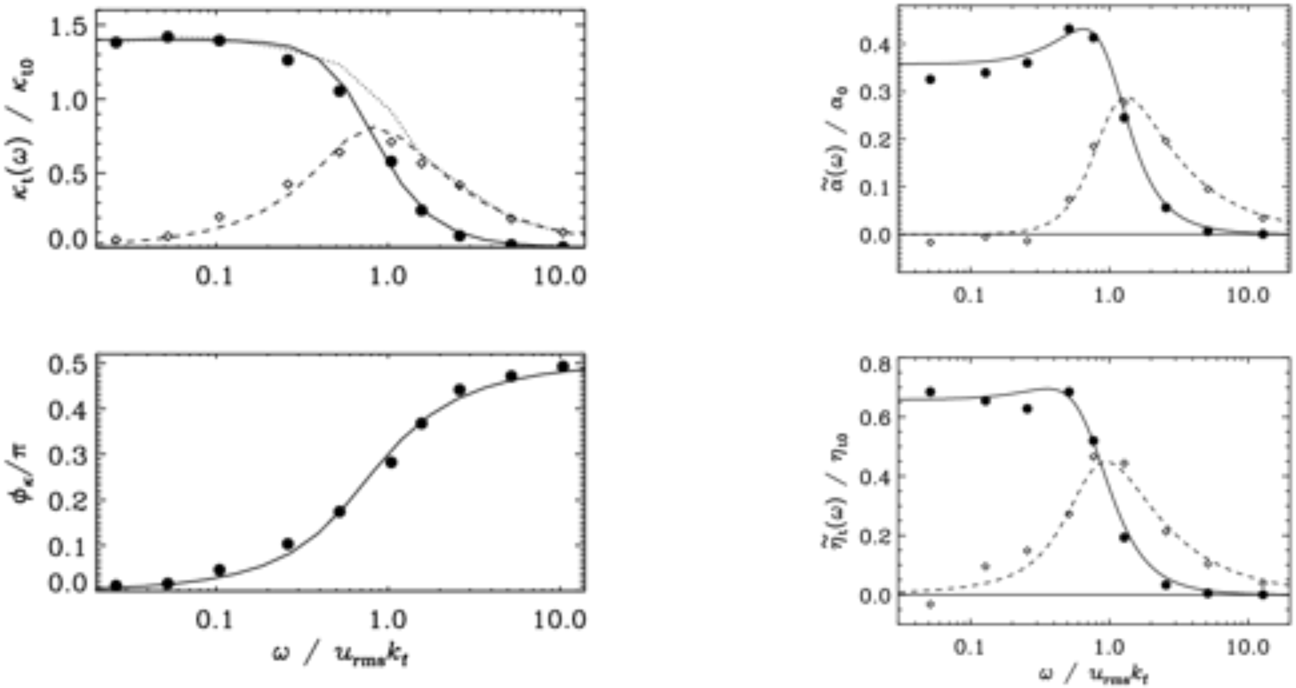
Unconfirmed for large k
Smagorinsky scaling $\sim 1/k^{-4/3}$

$$\kappa_{ij} = \kappa_t \delta_{ij} + \kappa_S S_{ij} + \kappa_A A_{ij} + \kappa_{SS} S_{ik} S_{kj} + \kappa_{AS} A_{ik} S_{kj}$$

Madarassy & Brandenburg,, (2010, PRE)

Nonlocality in time

$$\alpha\overline{\mathbf{B}} \text{ ® } \int \hat{\alpha}(t-t')\overline{\mathbf{B}}(t')\,\mathrm{d}t' \qquad \hat{\alpha}(t) = \int_{-\infty}^{\infty} \tilde{\alpha}(\omega) e^{-\mathrm{i}\omega t}\,\frac{\mathrm{d}\omega}{2\pi}$$



Hubbard & Brandenburg (2009, ApJ 706, 712)

$$\alpha(\omega) = \frac{1-\mathrm{i}\omega\tau}{(1-\mathrm{i}\omega\tau)^2+\omega_0^2\tau^2} \approx \frac{1}{1-\mathrm{i}\omega\tau} \quad \Leftrightarrow \quad \alpha(t) = e^{-t/\tau}\cos\omega_0 t \approx e^{-t/\tau}$$

Revisit Fickian diffusion

Passive scalar equation

$$\frac{\partial C}{\partial t} = -\nabla \times \left(\mathbf{U} C - \kappa \nabla C \right)$$

Averages and fluctuations

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{u}, \qquad C = \overline{C} + c$$

Averaged (mean-field) equation

$$\frac{\partial \overline{C}}{\partial t} = -\nabla \times \left(\overline{\mathbf{U}}\,\overline{C} + \overline{\mathsf{F}} - \kappa \nabla C \right)$$

$$\overline{\mathsf{F}} = -\kappa_{\mathrm{t}} \nabla \overline{C} \quad \text{or} \quad \overline{\mathsf{F}}_i = -\kappa_{ij} \nabla_j \overline{C}$$

→ **shortcomings: causality, nonlocality in space & time, & pumping**

Tau approximation: passive scalars

$$\overline{\mathbf{U}} = 0$$

$$\frac{\partial C}{\partial t} = -\mathbf{U} \times \nabla C$$ primitive eqn

$$\frac{\partial c}{\partial t} = -\mathbf{u} \times \nabla \overline{C} - \mathbf{u} \times \nabla c - \overline{\mathbf{u} \times \nabla c}$$ fluctuations

$$\frac{\partial \overline{\mathbf{u}c}}{\partial t} = -\overline{\mathbf{u}\mathbf{u}} \times \nabla \overline{C} - \overline{\mathbf{u}\mathbf{u} \times \nabla c}$$ Flux equation
→ triple moment

$$\frac{\partial \overline{\mathbf{u}c}}{\partial t} = -\overline{\mathbf{u}\mathbf{u}} \times \nabla \overline{C} - \frac{\overline{\mathbf{u}c}}{\tau}$$ Closure assumtion

System of mean field equations

mean concentration $$\frac{\partial \overline{C}}{\partial t} = -\nabla \cdot \overline{\mathsf{F}}$$

flux equation $$\frac{\partial \overline{\mathsf{F}}}{\partial t} = -\overline{\mathbf{u}\mathbf{u}} \cdot \nabla \overline{C} - \frac{\overline{\mathsf{F}}}{\tau}$$

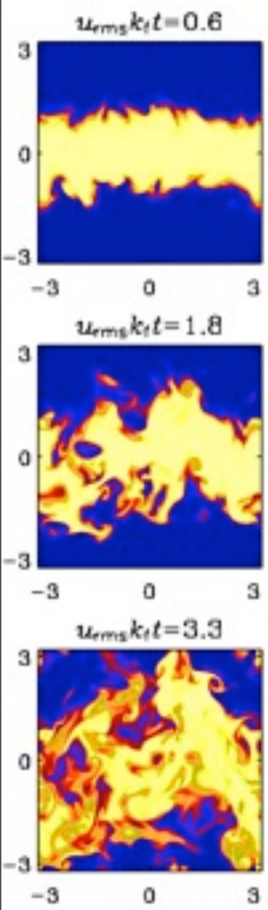
Damped wave equation, wave speed $\sqrt{\tfrac{1}{3}\overline{\mathbf{u}^2}}$

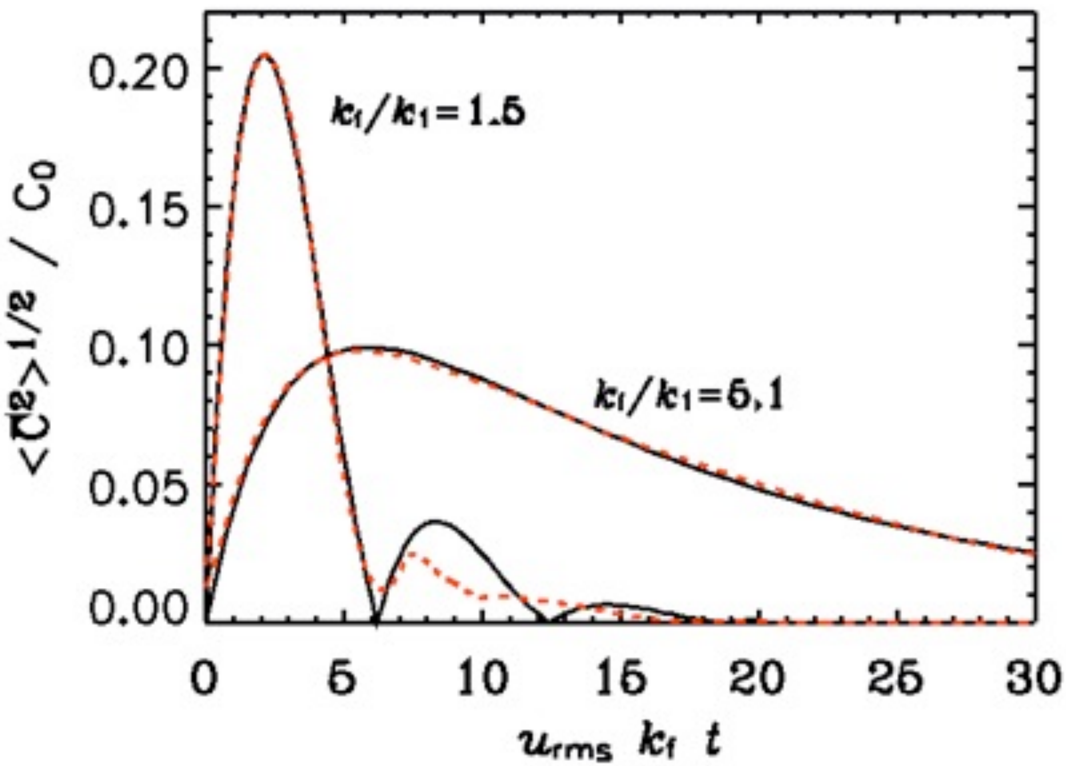
$$\frac{\partial^2 \overline{C}}{\partial t^2} + \frac{1}{\tau}\frac{\partial \overline{C}}{\partial t} = \tfrac{1}{3}\overline{\mathbf{u}^2}\nabla^2 \overline{C}$$ (causality!)

Blackman & Field (2003), Brandenburg et al. (2004)

Test: finite initial flux experiment

Initial state: $\overline{C} = 0$ but with $\overline{F} \neq 0$





black:
closure model

red:
turbulence sim.

→ direct evidence for oscillatory behavior!

Dispersion
Relation:
Oscillatory
for k_1/k_f<3

Nonlocality in time → memory effect

flux equation
$$\frac{\partial \overline{\mathcal{F}}}{\partial t} = -\overline{\mathbf{uu}} \cdot \nabla \overline{C} - \frac{\overline{\mathcal{F}}}{\tau}$$

solved by
$$\overline{\mathcal{F}}(t,\mathbf{x}) = -\int_0^t e^{-(t-t')/\tau}\,\mathbf{uu} \cdot \nabla \overline{C}(t',\mathbf{x})\,\mathrm{d}t'$$

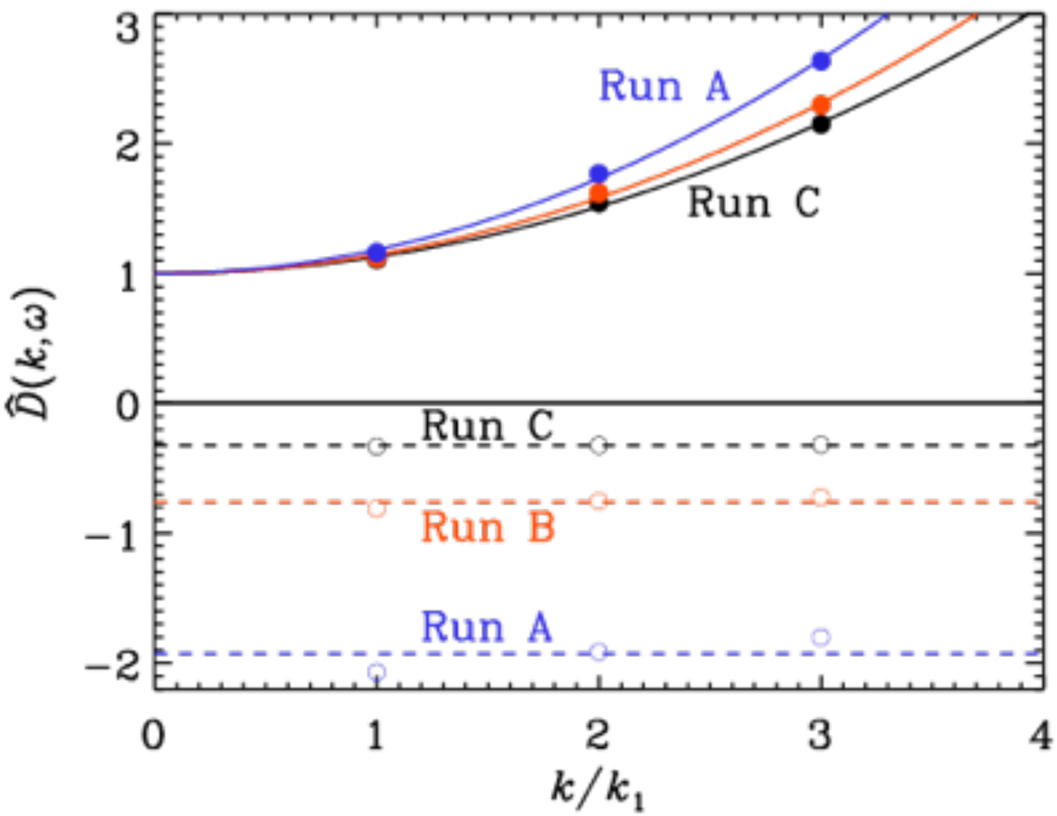
Convolution
in time
$$\overline{\mathcal{F}}_i(t,\mathbf{x}) = -\int_0^t \hat{\kappa}_{ij}(t-t',\mathbf{x})\nabla_j \overline{C}(t',\mathbf{x})\,\mathrm{d}t'$$

From multiplication to convolution

$$\overline{\mathcal{F}}_i = -\kappa_{ij} \cdot \nabla_j \overline{C} \quad \rightarrow \quad \overline{\mathcal{F}}_i = -\hat{\kappa}_{ij} \circ \nabla_j \overline{C}$$

Nonlocality in *x&t* via a PDE

$$\frac{1}{1 - i\omega\tau + k^2/k_\mathrm{f}^2} \qquad \left(1 - \tau\frac{\partial}{\partial t} + l^2\nabla^2\right)\overline{\mathbf{F}} = -\kappa_\mathrm{t}\nabla C$$

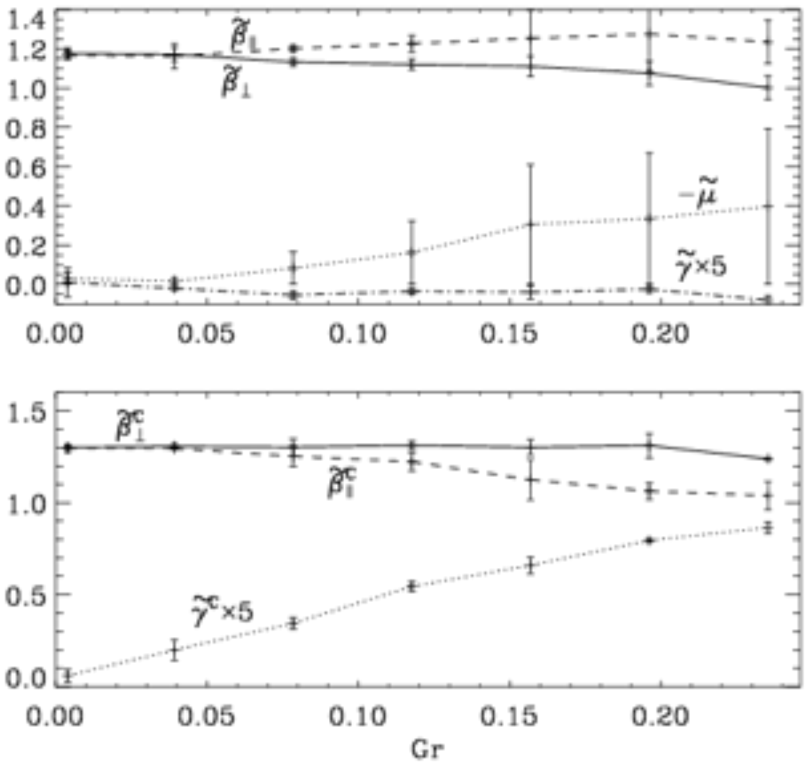


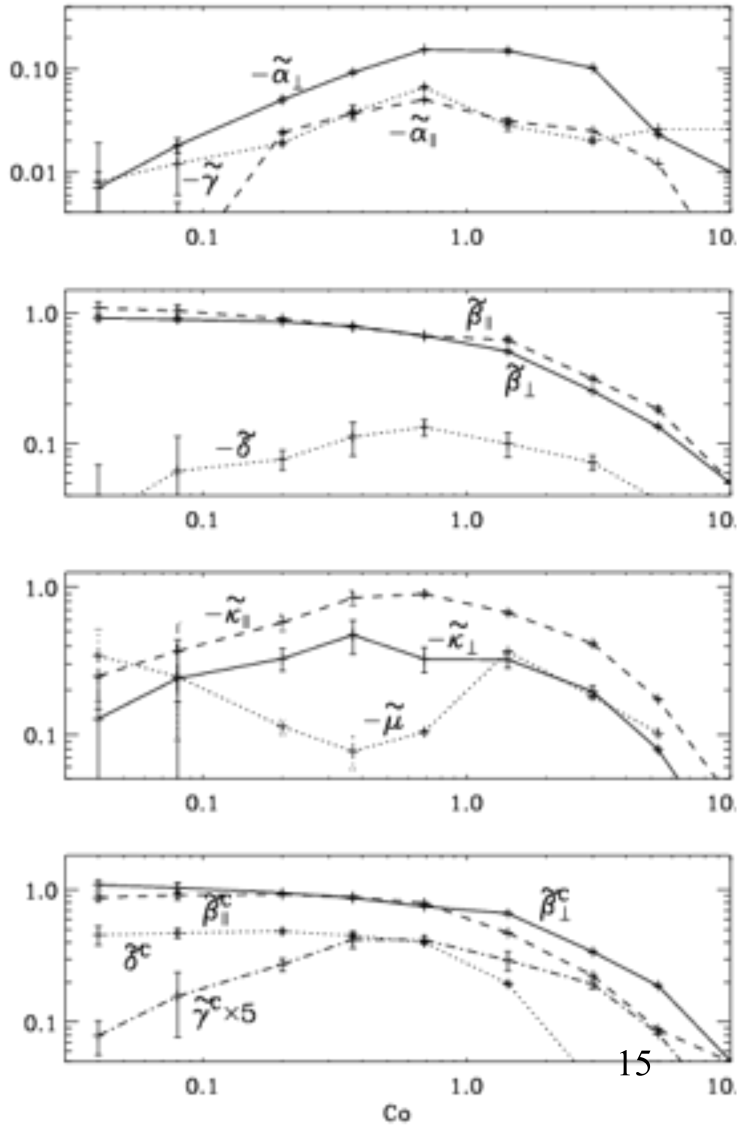
Rheinhard & Brandenburg (2012)

Turbulent pumping & anisotropy

One preferred direction e

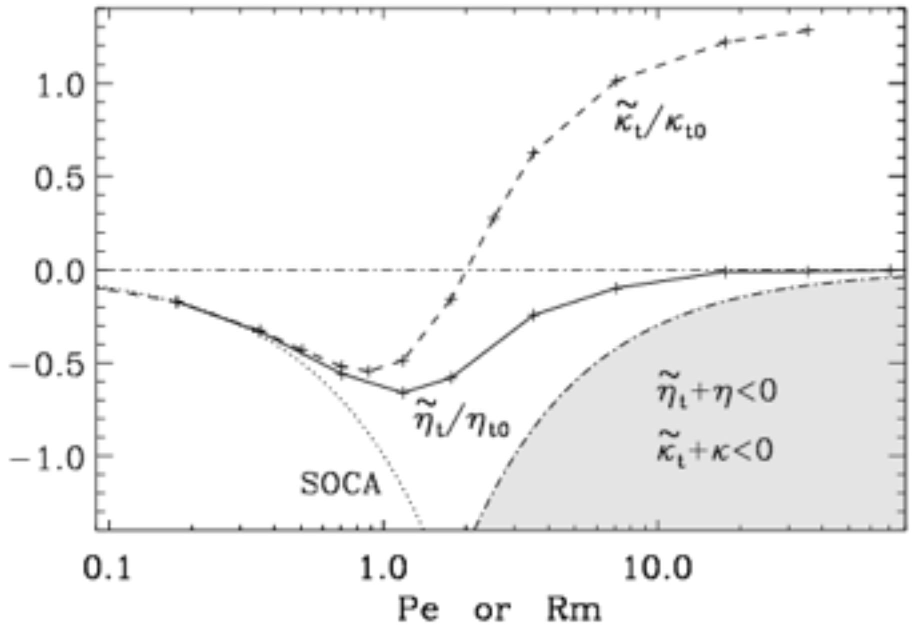
$$\mathbf{F} = -\gamma^{C}\overline{C}\,\hat{\mathbf{e}} - \beta_{\perp}^{C}\nabla\overline{C} - \left(\beta_{\parallel}^{C} - \beta_{\perp}^{C}\right)\left(\hat{\mathbf{e}}\times\nabla\overline{C}\right)\hat{\mathbf{e}} - \delta^{C}\,\hat{\mathbf{e}}\times\nabla\overline{C}$$

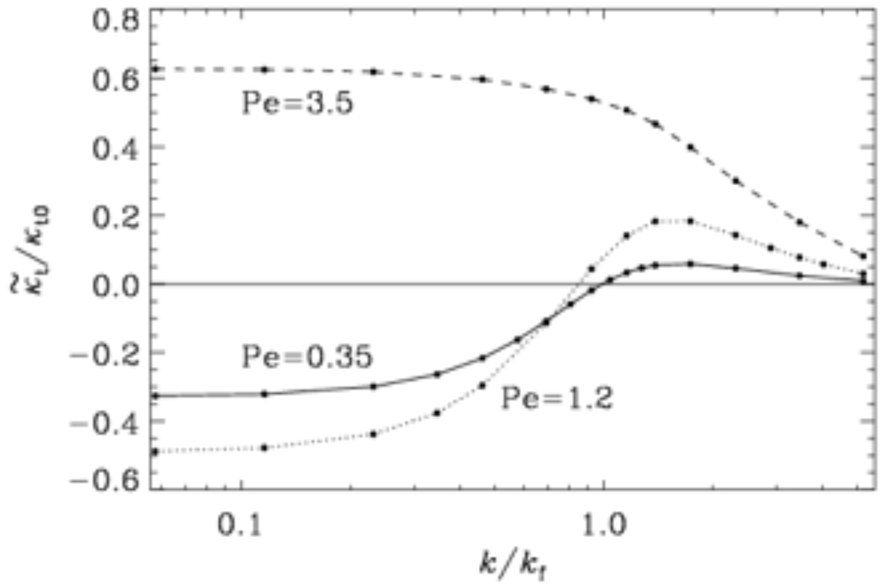




Potential flows (low Pe)

$$\mathbf{U} = \nabla \times \boldsymbol{\psi} - \nabla\phi \qquad \kappa_t = \frac{1}{3\kappa}\left(\psi^2 - \phi^2\right)$$





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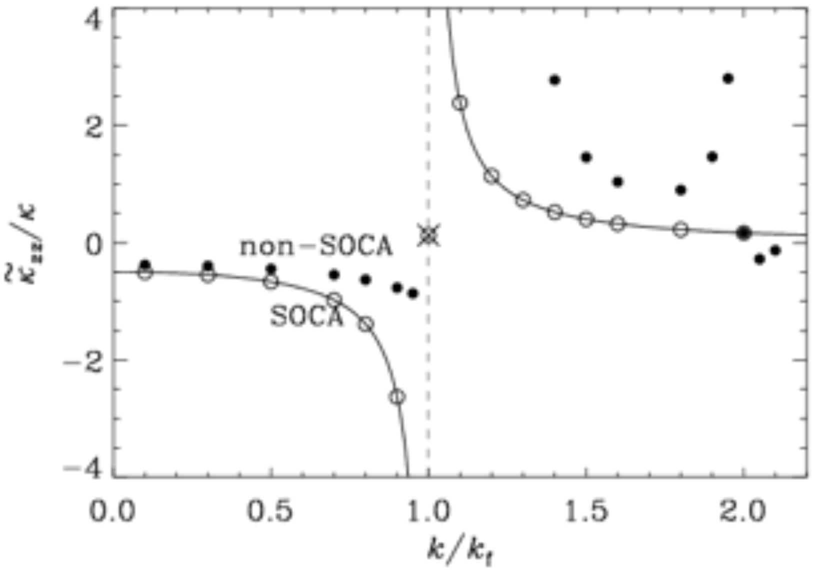
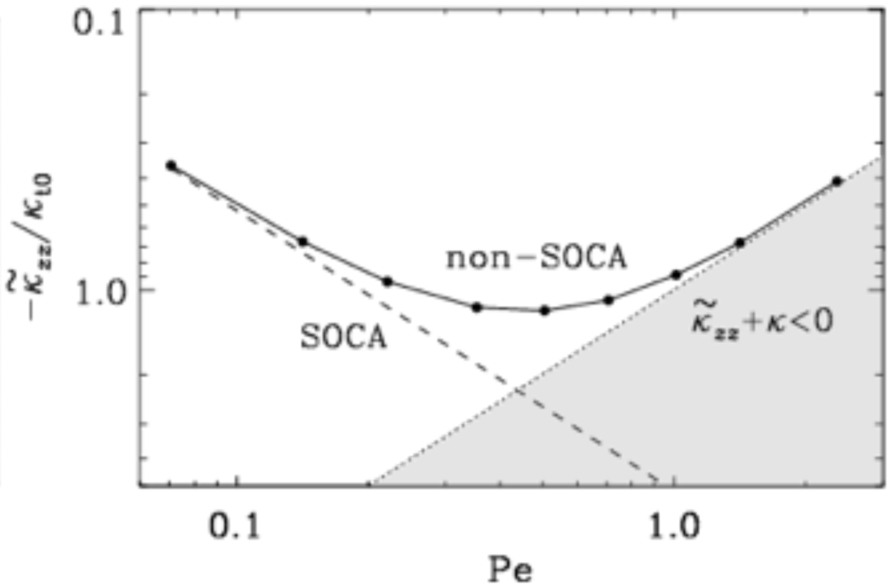
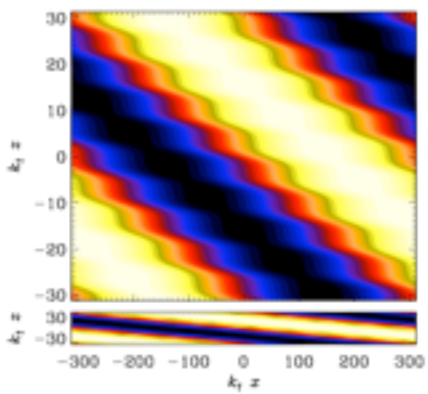
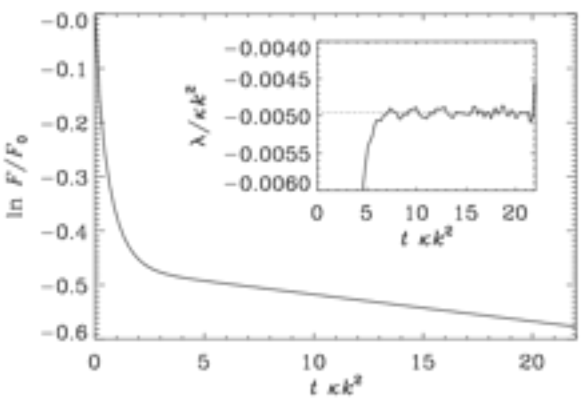
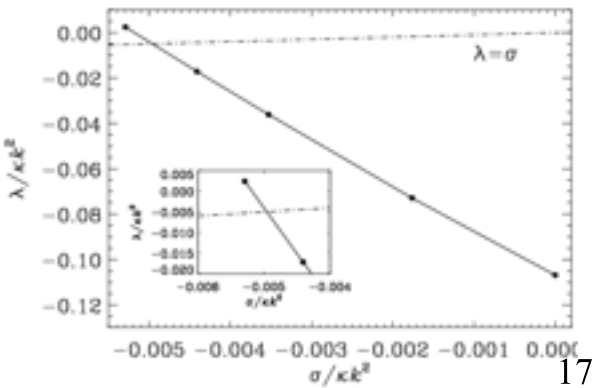
Potential flows











Broader context: mean-field concept

a effect, turbulent diffusivity, Yoshizawa effect, etc

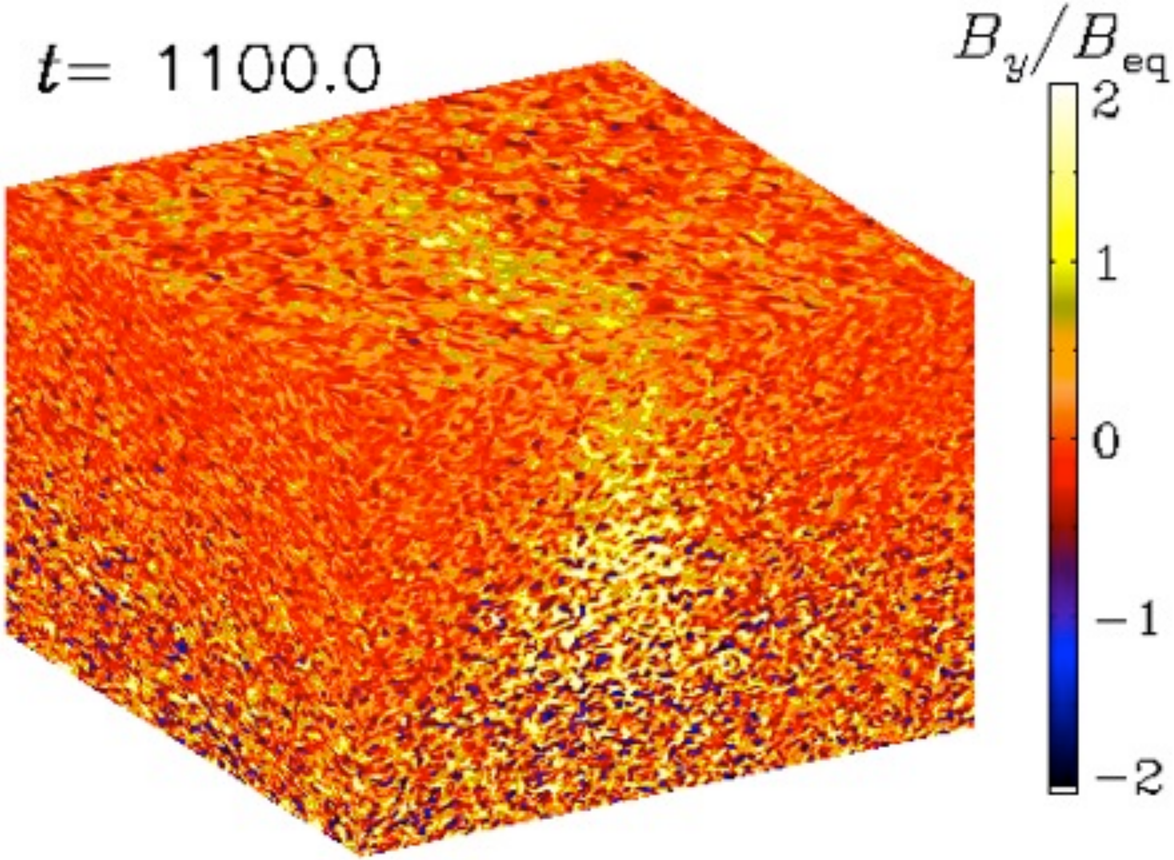
$$\overline{\mathbf{u}\times\mathbf{b}} = \alpha\overline{\mathbf{B}} - \eta_{\mathrm{t}}\overline{\mathbf{J}} + \ldots + \Xi\overline{\mathbf{U}} + \gamma\overline{\mathbf{W}}$$

Turbulent viscosity and other effects in momentum equation

$$\overline{u_i u_j} = \Lambda_{ijk}\overline{\Omega}_k + \lambda_{ijk}\overline{U}_k + \ldots - \nu_{\mathrm{t}}\left(\overline{U}_{i,j} + \overline{U}_{j,i}\right) +$$

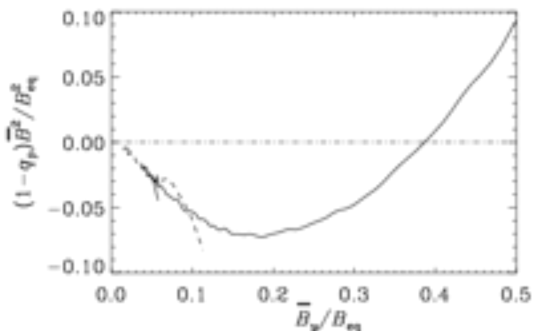
$$\ldots + \underbrace{\psi_{ijk}\overline{B}_k + \phi_{ijkl}\overline{B}_{k,l}}_{\text{Rheinhardt \& Brandenburg (2010)}} + \ldots + \underbrace{q_{\mathrm{s}}\overline{B}_i\overline{B}_j - \tfrac{1}{2}\delta_{ij}q_{\mathrm{p}}\overline{\mathbf{B}}^2}_{\text{Rädler (1974), Rüdiger (1974), Roberts \& Soward (1975)}} + \ldots$$

Negative effective magnetic pressure instability



Brandenburg et al. (2011, 2012), Kemel et al. (2012)

- Gas+turb. press equil.
- B increases
- Turb. press. Decreases
- Net effect?



And finally: dynamos in SNRs

$$\rho \frac{\partial \mathbf{U}}{\partial t} = -\nabla P + \tfrac{1}{c}\mathbf{J}\times\mathbf{B} + e(n_i - n_e)\mathbf{E} + \mathbf{F}_\nu$$

$$\nabla\times\mathbf{B} = \tfrac{4\pi}{c}\left(\mathbf{J} + \mathbf{J}_{\mathrm{cr}}\right) \qquad\qquad n_i + n_{cr} = n_e$$

$$\rho \frac{\partial \mathbf{U}}{\partial t} = -\nabla P + \left(\tfrac{1}{4\pi}\nabla\times\mathbf{B} - \tfrac{1}{c}\mathbf{J}_{\mathrm{cr}}\right)\times\mathbf{B} + e n_{\mathrm{cr}}\mathbf{U}\times\mathbf{B} + \mathbf{F}_\nu$$

To be solved with induction equation
and continuity equation, isothermal EOS

Introduces pseudoscalar

Ù $\times$**g** ® α effect in stars

$$\mathbf{J}_{\mathrm{cr}} \times \mathbf{B}_0 \text{ ® } \alpha \text{ effect}$$

α effect important for large-scale field in the Sun

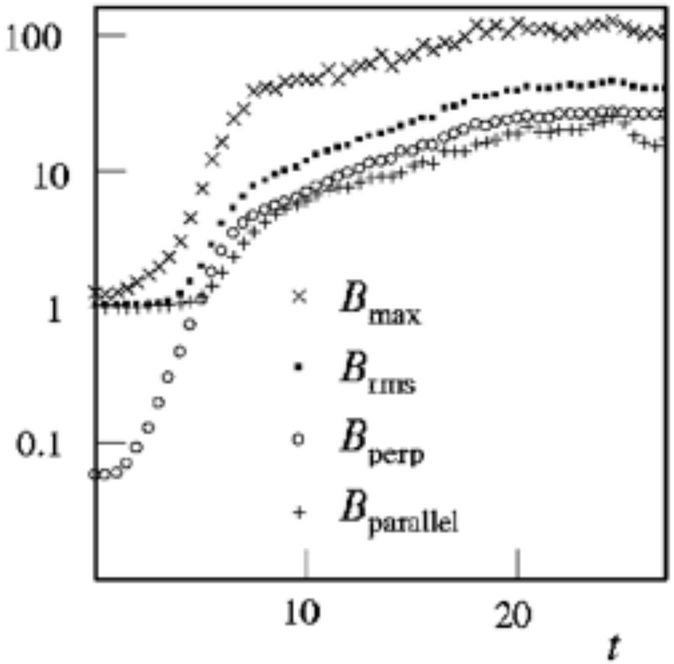
$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \overline{\mathbf{u} \times \mathbf{b}} - \overline{\mathbf{J}} / \sigma \right)$$

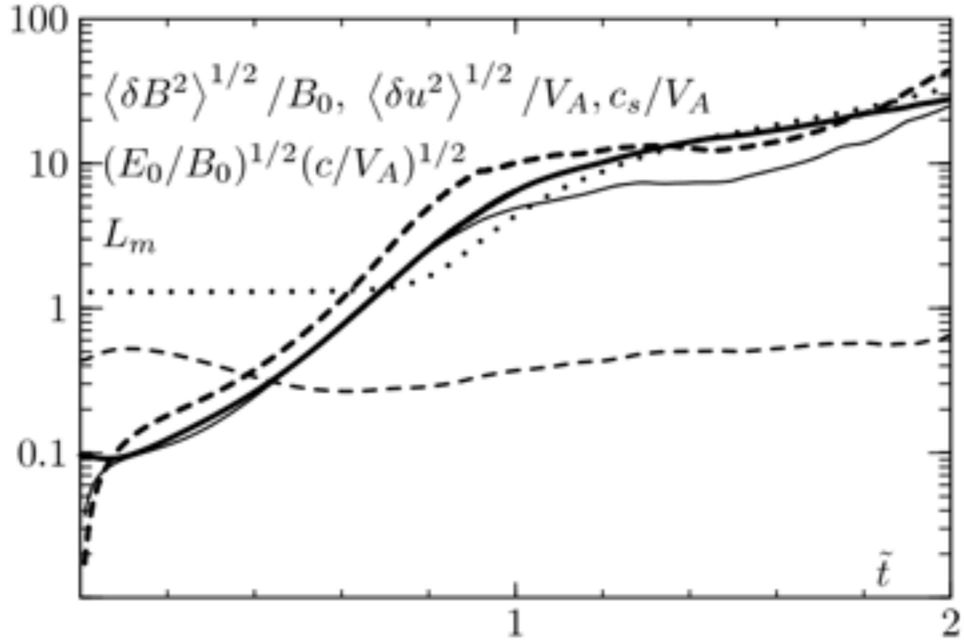
$$\overline{\mathsf{E}} \equiv \overline{\mathbf{u} \times \mathbf{b}} = \alpha \overline{\mathbf{B}} + \ldots$$

$$\overline{\mathsf{E}}_i = \alpha_{ij} \overline{B}_j - \eta_{ij} \overline{J}_j + \ldots$$

Bell instability

$$\gamma_B^2 = \left(\frac{4\pi}{c} \frac{J_{\mathrm{cr}}}{B} k_z - k^2 \right) v_A^2 \qquad \mathrm{J} = \frac{4\pi}{c} J_{\mathrm{cr}} / kB_0$$





Bell (2004): J=2

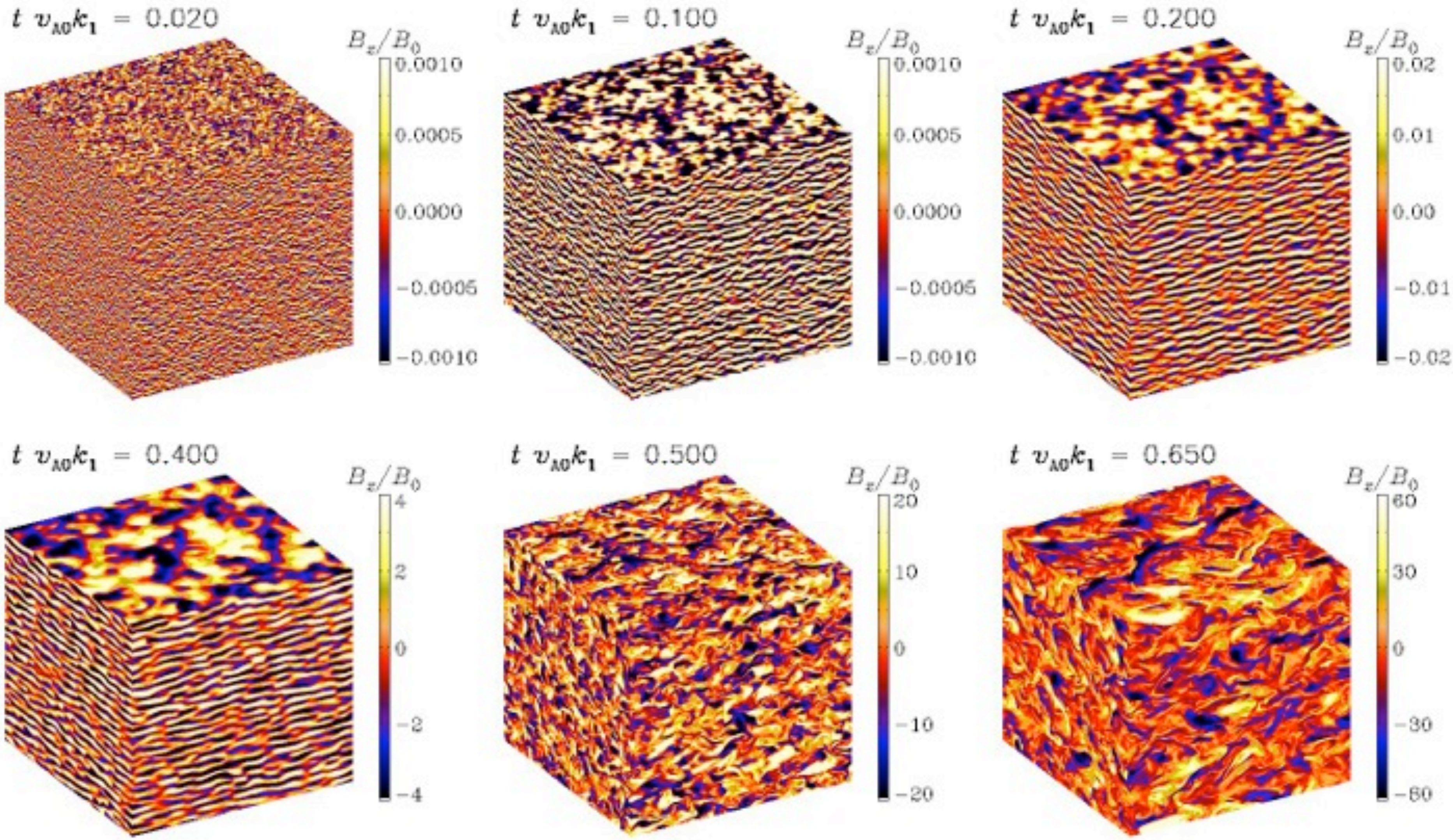
Zirakashvili et al (2008): J=16

Continued growth in both cases! → α effect important?

New simulations

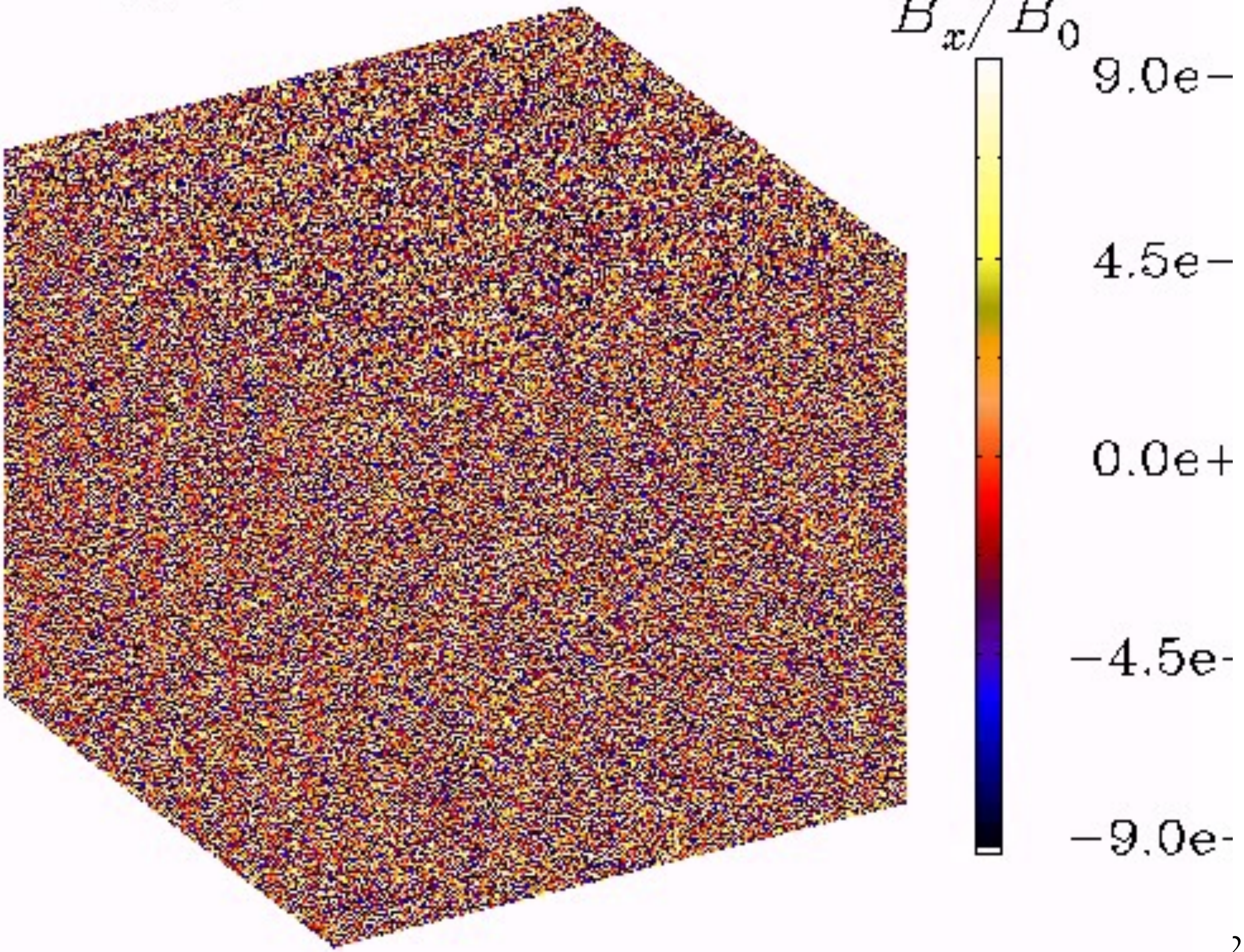
- 512^3 resolution, non-ideal (Re=Lu < 300)
- larger J parameter (80 and 800)
- most unstable k/k_1= 40 and 400 (unresolved)
- measure alpha and turbulent diff. tensor
- Related to earlier work by Bykov et al. (2011)

Bell instability → turbulence (J=80)

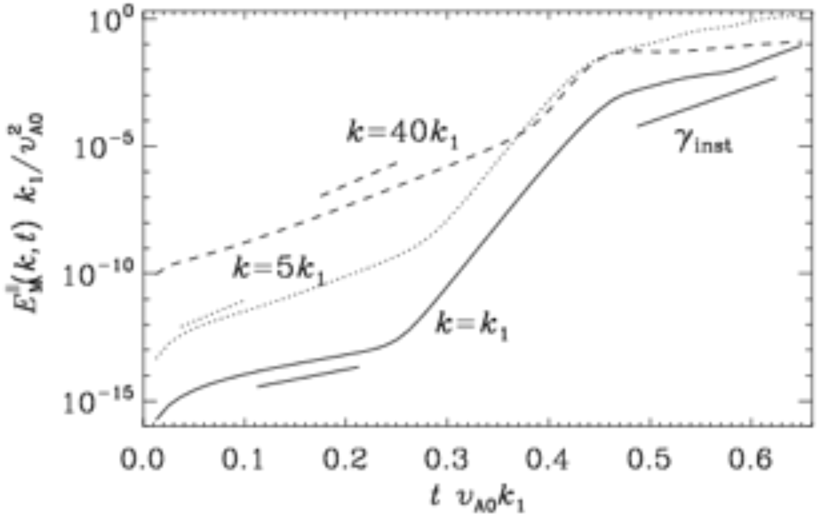


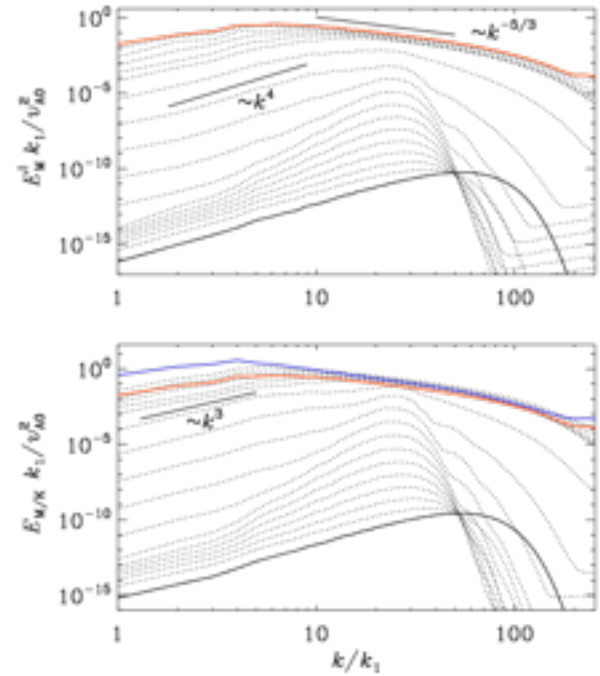
Animation

$t \ v_{A0} k_1 = 0.001$



3 stages





- Bell instability,small scale, k/k_1=40
- Accelerated large-scale growth
- Slow growth after initial saturation

Main points

- In MHD: $h_t = u_{rms}/3k_f$
- Moderate quenching at large R_m
- No approximation (such as in SOCA)
- Lorentzian in k
- Memory in time
- Transport even in homogeneous systems
- Nonlocality is of practical relevance
- Extension to Navier-Stokes, etc.